AS - 2828 | Model Answer

B.Sc. (Hon's) (Fifth Semester) Examination, 2013

Mathematics Paper-Second (Real Analysis)

1. (i) Convergent Sequence : A sequence $\{S_n\}$ is said to converge to a real number l if for every $\epsilon > 0$, there exists a positive number m such that

$$|S_n - l| < \epsilon \quad \forall\, n \geq m$$

This can be written as $\lim_{n\to\infty} S_n = l$

(ii) We know that sequence $\{S_n\}$ is said to be divergent to $+\infty$ if for each positive number G (however large), there exist a positive integer m such that $S_n > G \quad \forall\, n \geq m$

We have to show that $\langle 3^n \rangle$ diverges to $+\infty$

Let $G > 0$ and $S_n = 3^n$

$$S_n > G \Rightarrow 3^n > G \Rightarrow \log 3^n > \log G \Rightarrow n > \frac{\log G}{\log 3}$$

If we take $m > \frac{\log G}{\log 3}$ then

$$S_n > G \quad \forall\, n \geq m \Rightarrow \lim_{n\to\infty} S_n = \infty \Rightarrow \lim_{n\to\infty} 3^n = \infty$$

ie Sequence $\{3^n\}$ diverges to $+\infty$.

(iii) $\sum \frac{1}{2^n + x}$, $x \geq 0$

$$u_n = \frac{1}{2^n + x}, \quad u_{n+1} = \frac{1}{2^{n+1} + x}$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2^{n+1}+x}{2^n + x} = \lim_{n\to\infty} \frac{\cancel{2^n}\left(2 + \frac{x}{2^n}\right)}{\cancel{2^n}\left(1 + \frac{x}{2^n}\right)}$$

$$= \lim_{n\to\infty} \frac{2 + x/2^n}{1 + x/2^n}$$

$$= 2 > 1$$

By D'Alembert's Ratio test $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ if $l > 1$ series is Conv.

here $2 > 1$, Hence Given series $\sum \frac{1}{2^n + x}$, $x \geq 0$ is Convergent.

(2)

(iv) $\Sigma \frac{1}{n^n}$

$u_n = \frac{1}{n^n}$ Applying Cauchy's n^{th} Root test

$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty}\frac{1}{n} = 0 < 1$

If $l < 1$ series is Convergent hence
Given series $\Sigma \frac{1}{n^n}$ is Convergent.

(v) Let $P = \{a = x_0, x_1, x_2, ----, x_{r-1}, x_r, ---- x_n = b\}$
and $P' = \{a = x_0, x_1, x_2, -----, x_{r-1}, \xi_1, x_r, ---- x_n = b\}$
P' is refinement of P ($\because x_{r-1} < \xi_1 < x_r$ and ξ_1 is additional pt in P')

Let Interval	Infremum	Supremum
$[x_{r-1}, \xi_1]$	m_r'	M_r'
$[\xi_1, x_r]$	m_r''	M_r''
$[x_{r-1}, x_r]$	m_r	M_r

Clearly $m_r \le m_r'$, $m_r \le m_r''$
and $M_r \geq M_r'$, $M_r \geq M_r''$

Consider $L(P', f) - L(P, f)$
$= m_r'(\xi_1 - x_{r-1}) + m_r''(x_r - \xi_1) - m_r(x_r - x_{r-1})$
$= m_r'(\xi_1 - x_{r-1}) + m_r''(x_r - \xi_1) - m_r(\xi_1 - x_{r-1}) - m_r(x_r - \xi_1)$
$= (m_r' - m_r)(\xi_1 - x_{r-1}) + (m_r'' - m_r)(x_r - \xi_1) \geq 0$ ($\because m_r \le m_r', m_r \le m_r''$)
Hence $L(P, f) \le L(P', f)$

(VI) $\because f$ is a bounded function.
Let M, m are sup. and inf. of f in $[a, b]$. Let $\epsilon > 0$.
Let no. of points of discontinuity of f in $[a, b]$ is p. say $\{a_1, a_2, -- a_p\}$
Consider p non-overlapping intervals $[a_1', a_1'']$, $[a_2', a_2'']$ ------ $[a_p', a_p'']$ such that length of each interval $\le \frac{\epsilon}{2(M-m)}$
Now total Contribution to these intervals in $U(P, f) - L(P, f) < \frac{\epsilon}{2(M-m)} \cdot (M-m)$
ie $U(P, f) - L(P, f) < \epsilon/2$ —— (1) ($\because U(P, f) - L(P, f)$ of each sub. int $< (M-m)$)

Contribution to $U(P, f) - L(P, f)$ of rest $(p+1)$ intervals
$[a, a_1']$, $[a_1'', a_2']$, --- $[a_p'', b]$ is $\le \frac{\epsilon}{2(p+1)} \cdot (p+1)$ ($U(P, f) - L(P, f)$ of each $(p+1)$ interval $< \epsilon/2(p+1)$)
$= \epsilon/2$

Hence $\exists$ a partition $[a, b]$ such that its $U(P, f) - L(P, f) < \epsilon/2 + \epsilon/2$
ie $U(P, f) - L(P, f) < \epsilon$
$\Rightarrow f$ is integrable in $[a, b]$.

(VII) $\int_{-1}^{1} \frac{dx}{(2-x)\sqrt{1-x^2}}$

$= \lim_{\lambda \to 0+} \int_{-1+\lambda}^{0} \frac{dx}{(2-x)\sqrt{1-x^2}} + \lim_{\mu \to 0+} \int_{0}^{1-\mu} \frac{dx}{(2-x)\sqrt{1-x^2}}$

$= \lim_{\lambda \to 0+} \left[\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3}{2-x} - 2\right)\right]_{-1+\lambda}^{0} + \lim_{\mu \to 0+} \left[\frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3}{2-x} - 2\right)\right]_{0}^{1-\mu}$

$= \frac{1}{\sqrt{3}} \sin^{-1}\left(-\frac{1}{2}\right) - \lim_{\lambda \to 0+} \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3}{3-\lambda} - 2\right)$

$+ \lim_{\mu \to 0+} \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3}{1+\mu} - 2\right) - \frac{1}{\sqrt{3}} \sin^{-1}\left(-\frac{1}{2}\right)$

$= -\frac{1}{\sqrt{3}} \sin^{-1}(-1) + \frac{1}{\sqrt{3}} \sin^{-1}(1)$

$= -\frac{1}{\sqrt{3}}\left(-\frac{\pi}{2}\right) + \frac{1}{\sqrt{3}} \cdot \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$ (finite)

$\int \frac{dx}{(2-x)\sqrt{1-x^2}}$ Put $2-x = 1/t$, $dx = 1/t^2 \, dt$

$= \int \frac{1}{t^2} \cdot \frac{t}{\sqrt{1-(2-1/t)^2}} dt$

$= \int \frac{dt}{\sqrt{-3t^2+4t-1}} = \frac{1}{\sqrt{3}} \int \frac{dt}{\sqrt{(1/3)^2 - (t-2/3)^2}}$

$= \frac{1}{\sqrt{3}} \sin^{-1}\left(\frac{3}{2-x} - 2\right)$

Hence Given Improper integral $\int_{-1}^{1} \frac{dx}{(2-x)\sqrt{1-x^2}}$ is <u>Convergent</u>

(VIII) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\int_{0}^{\infty} \frac{1}{1+x^2} dx$

$= 2 \lim_{X \to \infty} \int_{0}^{X} \frac{1}{1+x^2} dx$

$= 2 \lim_{X \to \infty} \left[\tan^{-1} x\right]_{0}^{X} = 2 \lim_{X \to \infty} (\tan^{-1} X - \tan^{-1} 0)$

$= 2 \cdot \frac{\pi}{2} = \pi$ (finite)

Hence Improper Integral $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ is <u>Convergent</u>

<u>2. (a)</u> Let $\{S_n\}$ be a bounded sequence and l be its unique limit point. Now we have to show that $\{S_n\}$ converges to l.

Let $\epsilon > 0$. $\because$ l is only limit point of $\{S_n\}$.

$\therefore$ then any neighbourhood of l ie $(l-\epsilon, l+\epsilon)$ contains infinite no. of elements of $\{S_n\}$ and there will be only finite elements of $\{S_n\}$, outside of $(l-\epsilon, l+\epsilon)$.

(For if infinite no. of elements exit outside $(l-\epsilon, l+\epsilon)$ then there may be another limit point).

ie $(l-\epsilon, l+\epsilon)$ contains infinite no. of elements of sequence $\{S_n\}$ except some finite elements, say m elements.

ie $S_n \in (l-\epsilon, l+\epsilon) \quad \forall n \geq m$

$\Rightarrow |S_n - l| < \epsilon \quad \forall n \geq m$

$\Rightarrow$ Sequence $\{S_n\}$ converges to l.

2(b) By Cauchy's First theorem on limits

if $\lim\limits_{n\to\infty} f_n = l$ then $\lim\limits_{n\to\infty} \dfrac{f_1+f_2+\cdots f_n}{n} = l$

Take $f_n = n^{1/n}$ We know that $\lim\limits_{n\to\infty} n^{1/n} = 1$

then $\lim\limits_{n\to\infty} \dfrac{1+2^{1/2}+3^{1/2}+\cdots n^{1/n}}{n} = 1$

ie $\lim\limits_{n\to\infty} \frac{1}{n}\left[1+2^{1/2}+3^{1/2}+\cdots n^{1/n}\right] = 1$.

3. <u>Cauchy's General Principle of Convergence for Sequences</u>

<u>Statement:</u> A necessary and sufficient condition for a sequence $\{S_n\}$ to be convergent is that to each $\epsilon>0$ there corresponds a positive integer m such that

$$|S_{n+p}-S_n| < \epsilon \quad \forall\ n\geq m,\ p\geq 0$$

<u>Proof</u> <u>Condition is Necessary</u>: Let Sequence $\{S_n\}$ be convergent then there exists l such that $\lim\limits_{n\to\infty} S_n = l$

then for $\epsilon>0$, there exists $m\in\mathbb{N}$ such that

$$|S_n - l| < \epsilon/2 \quad \forall\ n\geq m \quad \text{———} \ (1)$$

If $p\geq 0$ then $n\geq m \Rightarrow n+p\geq m$

$$|S_{n+p}-l| < \epsilon/2 \quad \forall\ n\geq m,\ p\geq 0 \quad \text{———} \ (2)$$

Consider
$$\begin{aligned}
|S_{n+p}-S_n| &= |S_{n+p}-l+l-S_n| \\
&\leq |S_{n+p}-l| + |l-S_n| \\
&< \epsilon/2 + \epsilon/2 = \epsilon \quad \forall\ n\geq m,\ p\geq 0 \quad \text{(from (1) and (2))}
\end{aligned}$$

Hence we have $|S_{n+p}-S_n| < \epsilon \ \forall\ n\geq m,\ p\geq 0$

Showing that Condition is Necessary.

<u>Condition is Sufficient</u>: Given condition $|S_{n+p}-S_n| < \epsilon \ \forall\ n\geq m,\ p\geq 0$ is true $\forall\ \epsilon$. Take $\epsilon = 1$ then $\exists\ m_0>0$

such that $|S_{n+p}-S_n| < 1 \quad \forall\ n\geq m_0,\ \forall\ p\geq 0$

If $n=m_0$ $\quad |S_{m_0+p}-S_{m_0}| < 1 \quad \forall\ p\geq 0$

$$S_{m_0}-1 < S_{m_0+p} < S_{m_0}+1 \quad \forall\ p\geq 0$$

Let $g = \min\{S_1, S_2 \cdots S_{m_0-1},\ S_{m_0}-1\}$

$G = \max\{S_1, S_2, \cdots S_{m_0-1},\ S_{m_0}+1\}$

then $g \leq S_n \leq G \quad \forall\ n$ Showing that S_n is bounded sequence. Then by Bolzano Weierstrass Theorem for Sequences, S_n has

Now we have to show that $\lim_{n\to\infty} S_n = l$

From Given Condition $|S_{n+p} - S_n| < \epsilon/3 \quad \forall\ n \geq m,\ p \geq 0$

Take $n = m$ $\quad |S_{m+p} - S_m| < \epsilon/3 \quad \forall\ p \geq 0$ ——— (3)

$\because$ l is limit point of sequence $\{S_n\}$ then neighbourhood of l say $(l - \epsilon/3, l + \epsilon/3)$ contains infinite elements of sequene S_n. Then $\exists\ m_1 > m$ such that $S_{m_1} \in (l - \epsilon/3, l + \epsilon/3)$

$\Rightarrow \quad |S_{m_1} - l| < \epsilon/3$ ——— (4)

$\because m_1 > m$ then from (3) we have $|S_{m_1} - S_m| < \epsilon/3$ ——— (5)

Consider $|S_{m+p} - l| = |S_{m+p} - S_m + S_m - S_{m_1} + S_{m_1} - l|$

$\leq |S_{m+p} - S_m| + |S_m - S_{m_1}| + |S_{m_1} - l|$

Using (3), (4), (5) we get

$|S_{m+p} - l| < \epsilon/3 + \epsilon/3 + \epsilon/3 \quad \forall\ p \geq 0$

$\Rightarrow \quad |S_{m+p} - l| < \epsilon \quad \forall\ p \geq 0$

$\Rightarrow \quad |S_n - l| < \epsilon \quad \forall\ n \geq m$ Showing that $\{S_n\}$ Converges to l.

4.

Given series $1 + \frac{x}{1!} + \frac{2^2 \cdot x^2}{2!} + \frac{3^3 \cdot x^3}{3!} + \cdots$

n^{th} term of series $u_n = \frac{n^n \cdot x^n}{n!}$, $\quad u_{n+1} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!}$

By D'Alembert's Ratio test

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^n \cdot x^n \quad (n+1)!}{n! \ (n+1)^{n+1} \cdot x^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{x} \frac{\cancel{(n+1)}}{\cancel{(n+1)}} \cdot \left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty} \frac{1}{x} \cdot \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{x} \cdot \frac{1}{\lim_{n\to\infty} (1+\frac{1}{n})^n} = \frac{1}{ex}$$

Now series u_n is Convergent if $\frac{1}{ex} > 1 \Rightarrow x < 1/e$

divergent if $\frac{1}{ex} < 1 \Rightarrow x > 1/e$

test fails if $\frac{1}{ex} = 1 \Rightarrow x = 1/e$

at $x = \frac{1}{e}$ $\quad \frac{u_n}{u_{n+1}} = e \cdot \left(\frac{n}{n+1}\right)^n$

By Logaritmic test $\quad \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log_e \left\{ e \cdot \left(\frac{n}{n+1}\right)^n \right\}$

$$= \lim_{n\to\infty} n \cdot \left[\log_e e + n \log \frac{n}{n+1}\right]$$

$$= \lim_{n\to\infty} n\left[1 - n\log\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} n\left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= \lim_{n\to\infty} n\left(\frac{1}{2n} - \frac{1}{3n^2} \cdots\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} - \frac{1}{3n} \cdots = \frac{1}{2} < 1$$

Hence at $x = 1/e$ series Σu_n is divergent.

Given series Σu_n is Convergent if $x < 1/e$
divergent if $x \geq 1/e$

5. (a) Raabe's Test

Statement: If Σu_n is a positive term series, such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, then the series

i) Converges if $l > 1$
ii) diverges if $l < 1$
iii) test fails if $l = 1$

Proof Case-1 $l > 1$. Take $\epsilon > 0$ such that $l - \epsilon > 1$ say $l - \epsilon = \alpha > 1$

$\because \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then $\exists\ m > 0$ such that

$$\left| n\left(\frac{u_n}{u_{n+1}} - 1\right) - l \right| < \epsilon$$

$$\Rightarrow l - \epsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \epsilon$$

$\Rightarrow \alpha < \frac{n u_n - n u_{n+1}}{u_{n+1}} \Rightarrow \alpha u_{n+1} < n u_n - n u_{n+1}$

$\Rightarrow n u_n - (n+1) u_{n+1} > (\alpha - 1) u_{n+1} \quad \forall\ n \geq m$

Put
$n = m$: $m u_m - \cancel{(m+1) u_{m+1}} > (\alpha-1) u_{m+1}$
$n = m+1$: $\cancel{(m+1) u_{m+1}} - \cancel{(m+2) u_{m+2}} > (\alpha-1) u_{m+2}$
$n = m+2$: $\cancel{(m+2) u_{m+2}} - \cancel{(m+3) u_{m+3}} > (\alpha-1) u_{m+3}$
$\vdots$
$n = n-1$: $\cancel{(n-1) u_{n-1}} - n u_n > (\alpha-1) u_n$

Adding these we have $m u_m - n u_n > (\alpha-1)(u_{m+1} + u_{m+2} \cdots u_n)$

$$m u_m - n u_n > (\alpha-1)(S_n - S_m)$$

$\Rightarrow (\alpha-1)(S_n - S_m) < m u_m \quad \forall\ n \geq m$

$\Rightarrow S_n < S_m + \frac{m}{\alpha-1} u_m \quad \forall\ n \geq m$

$\because m$ is fixed then $S_m + \frac{m}{\alpha-1} u_m$ is fixed number.

Showing that sequence of partial sum S_n of given series Σu_n is bounded above hence Series Σu_n is convergent.

Case-II $l < 1$, take $\epsilon > 0$ such that $l + \epsilon < 1$

$\because \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$, therefore $\exists\ m > 0$ such that

$$\left| n\left(\frac{u_n}{u_{n+1}} - 1\right) - l \right| < \epsilon \quad \forall\ n \geq m$$

$$l - \epsilon < n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \epsilon < 1$$

$$\Rightarrow n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1 \quad \Rightarrow \frac{u_n}{u_{n+1}} < \left(1 + \frac{1}{n}\right) \quad \Rightarrow \frac{u_{n+1}}{u_n} > \frac{n}{n+1}$$

If we take $v_n = \frac{1}{n}$ then $\frac{u_{n+1}}{u_n} > \frac{v_{n+1}}{v_n}$

then by Comparison test of second type $\because v_n = \sum \frac{1}{n}$ is divergent

$\therefore \sum u_n$ will be also divergent.

Case-III Consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n(\log n)^2}$

$\sum \frac{1}{n}$ is divergent but $\sum \frac{1}{n(\log n)^2}$ is convergent.

but $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$ for both the series.

if $u_n = \frac{1}{n}$, $u_{n+1} = \frac{1}{n+1}$ $\quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{n+1}{n} - 1\right) = 1$

if $u_n = \frac{1}{n(\log n)^2}$ $\quad \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{(n+1)\{\log(n+1)\}^2}{n\{\log n\}^2} - 1\right) = 1$

(b) Given series $u_n = \sum \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$

Applying Cauchy's Root test

$$\lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}\right)^{1/n} = \lim_{n\to\infty} \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{1/2}}$$

$$= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{1 + \sqrt{n}}\right)^{\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\sqrt{n}}} = \frac{1}{e} < 1$$

$\because \lim_{n\to\infty} u_n^{1/n} = \frac{1}{e} < 1$

$\therefore$ Given series $\sum u_n = \left(1 + \frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ is Convergent

6 $\quad f(x) = \begin{cases} x + x^2 & x \in Q \\ x^2 + x^3 & x \notin Q \end{cases}$

Consider $(x + x^2) - (x^2 + x^3) = x(1 - x^2)$
$= x(1+x)(1-x)$

$\Rightarrow (x + x^2) - (x^2 + x^3) > 0$ if $0 < x < 1$
< 0 if $1 < x < 2$

$0 \quad 1 \quad 2$

$0 < x < 1 \Rightarrow 1 - x > 0$

$1 < x < 2 \Rightarrow 1 - x < 0$

x and $1 + x$ is +ive $\forall\ 0 < x < 2$

If M_r, m_r be the supremum and infremum of $f(x)$ in I_r ie r^{th} sub interval of any partition of $[0,2]$.

Then $\because$ $(x+x^2) - (x^2+x^3) > 0$ for $0 < x < 1$

Hence in $0 < x < 1$ $M_r = (x+x^2)$ and $m_r = (x^2+x^3)$

and $(x+x^2) - (x^2+x^3) < 0$ for $\phi < x < 2$

Hence in $1 < x < 2$ $M_r = (x^2+x^3)$ and $m_r = (x+x^2)$

Thus $M_r = \begin{cases} x+x^2 & 0<x<1 \\ x^2+x^3 & 1<x<2 \end{cases}$, $m_r = \begin{cases} x^2+x^3 & 0<x<1 \\ x+x^2 & 1<x<2 \end{cases}$

$$\underline{\int_0^2} f(x)\,dx = \underline{\int_0^1} f(x)dx + \underline{\int_1^2} f(x)\,dx$$

$$= \int_0^1 (x^2+x^3)\,dx + \int_1^2 (x+x^2)\,dx$$

$$= \left[\frac{x^3}{3}+\frac{x^4}{4}\right]_0^1 + \left[\frac{x^2}{2}+\frac{x^3}{3}\right]_1^2 = \left(\frac{1}{3}+\frac{1}{4}\right) + \left(2+\frac{8}{3}\right) - \left(\frac{1}{2}+\frac{1}{3}\right)$$

$$= \frac{53}{12}$$

and $$\overline{\int_0^2} f(x)\,dx = \overline{\int_0^1} f(x)\,dx + \overline{\int_1^2} f(x)\,dx$$

$$= \int_0^1 (x+x^2)\,dx + \int_1^2 (x^2+x^3)\,dx$$

$$= \left[\frac{x^2}{2}+\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}+\frac{x^4}{4}\right]_1^2$$

$$= \left(\frac{1}{2}+\frac{1}{3}\right) + \left(\frac{8}{3}+4\right) - \left(\frac{1}{3}+\frac{1}{4}\right) = \frac{83}{12}$$

$\because$ $\underline{\int_0^2} f(x)\,dx \neq \overline{\int_0^2} f(x)\,dx$ Hence f is Not Riemann Integrable in $[0,2]$

ie $f \notin R[0,2]$.

7. Generalised First Mean Value Theorem:

Statement: If f and g are integrable on $[a,b]$ and g keeps the same sign in $[a,b]$ then there exists a number μ lying between bounds of f such that $\int_a^b fg\,dx = \mu \int_a^b g\,dx$

Proof: Let m, M be the bounds of f $\forall x \in [a,b]$

then $m \le f(x) \le M$ ——— (1)

Take g is +ive over $[a,b]$

$\Rightarrow$ $m g(x) \le f(x)\, g(x) \le M g(x)$

f and g are integrable on $[a,b]$

then $m\int_a^b g(x)dx \le \int_a^b f(x)\,g(x)\,dx \le M\int_a^b g(x)\,dx$ if $b \geq a$

$m\int_a^b g(x)dx \geq \int_a^b f(x)\,g(x)\,dx \geq M\int_a^b g(x)\,dx$ if $a \geq b$

From above 2 equations we see, $\exists$ a number μ, $(m < \mu < M)$ such that

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$$

or $\int_a^b fg\, dx = \mu \int_a^b g\, dx$

Take $g(x)$ is -ive in $[a,b]$ then from ①

$$m\, g(x) \geq f(x)\, g(x) \geq M\, g(x)$$

Again f and g are integrable in $[a,b]$

$$m \int_a^b g(x)\, dx \geq \int_a^b f(x)\, g(x)\, dx \geq M \int_a^b g(x)\, dx \quad \text{if } b \geq a$$

$$m \int_a^b g(x)\, dx \leq \int_a^b f(x)\, g(x)\, dx \leq M \int_a^b g(x)\, dx \quad \text{if } a \geq b$$

from above 2 equations we see $\exists\, \mu : (M < \mu < m)$ such that

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$$

$$\int_a^b fg\, dx = \mu \int_a^b g\, dx$$

So in both cases when $g(x)$ is positive or negative we get μ such that $\int_a^b fg\, dx = \mu \int_a^b g\, dx$

8 $\int_0^1 x^{m-1} (1-x)^{n-1} \log x \, dx$

$\because$ in $[0,1]$ $\log x$ is negative hence we consider

positive integrand as $\int_0^1 x^{m-1} (1-x)^{n-1} (-\log x)\, dx$

$$I = \int_0^1 x^{m-1} (1-x)^{n-1} \log \frac{1}{x}\, dx$$

take $0 < \frac{1}{2} < 1$

$$I = \int_0^{1/2} x^{m-1} (1-x)^{n-1} \log \frac{1}{x}\, dx + \int_{1/2}^1 x^{m-1} (1-x)^{n-1} \log \frac{1}{x}\, dx$$

Convergence at $x = 0$

$$\int_0^{1/2} x^{m-1} (1-x)^{n-1} \log \frac{1}{x}\, dx$$

If $m-1 > 0$ this integral will be proper integral

if $m-1 \leq 0$ then $x=0$ will be point of infinite discontinuity

Consider $f(x) = x^{m-1} (1-x)^{n-1} \log \frac{1}{x} = \dfrac{(1-x)^{n-1} \log \frac{1}{x}}{x^{1-m}}$

take $g(x) = \dfrac{1}{x^p}$

then $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to 0} \dfrac{(1-x)^{n-1} \log \frac{1}{x}}{x^{1-m}} \cdot x^p = \lim\limits_{x\to 0} x^{p+m-1} (1-x)^{n-1} \left(\log \frac{1}{x}\right)$

this limit tends to zero if $p+m-1 > 0$

$m > 1-p > 0$

Now $\int_0^{1/2} g(x)\, dx = \int_0^{1/2} \frac{1}{x^p}\, dx$ will be convergent if $p < 1$

$p < 1 \Rightarrow 1-p > 0$ Hence $m > 1-p > 0$ is true

Convergence at $x=1$

$$\int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\log 1/x$$

for $n \geq 0$ integral will be proper integral

for $n < 0$ $x=1$ will be point of infinite discontinuity.

for $n<0$

$$f(x) = x^{m-1}(1-x)^{n-1}\log 1/x = \frac{x^{m-1}\log 1/x}{(1-x)^{1-n}}$$

take $g(x) = \frac{1}{(1-x)^{q}}$

Now $$\lim_{x\to 1}\frac{f(x)}{g(x)} = \lim_{x\to 1}\frac{x^{m-1}\log 1/x\,(1-x)^{q}}{(1-x)^{1-n}}$$

$$= \lim_{x\to 1}\frac{x^{m-1}\log 1/x}{(1-x)^{1-n-q}}$$ this limit will be finite and tends to one if $1-n-q<1$

$\Rightarrow -n-q<0 \Rightarrow n>-q$

Now $\int_{1/2}^{1} g(x)\,dx = \int_{1/2}^{1}\frac{1}{(1-x)^{q}}\,dx$ is Convergent if $q<1$
$\Rightarrow -q>-1$

Hence Condition will be $n>-q>-1$

ie $\boxed{n>-1}$

Hence required Condition on m and n will be

$m>0$ and $n>-1$ for the Convergence of given improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}\log x\,dx$.

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